1. REGULAR INTEREST

1.1 General Introduction: Percentages

Percentages are often used only when the given rates are constant, that is, unchanging over time. In such cases we say that a percentage is simply a way of recording a fraction whose denominator is 100.

However, many practical problems are in one way or another connected with time: demographics calculates the annual percentage growth of a population, commerce the monthly percentage growth of turnover, nuclear physics the daily, monthly or yearly percent reduction in radioactive materials, and so forth.

Such problems investigate the change over time of a given unit (occasionally termed *population*). This change may be either discrete or continuous. Discrete change is understood as a process in which there occurs a sudden change from one population value to another. The time that passes between changes is variously called the *transition*, *recalculation*, or *conversion period,* or simply *period*. We shall most commonly employ the latter term.

The rate of population change, in the case of both discrete and continuous growth, is shown by a coefficient of the speed of change known as the *percentage rate*. Because the percentage rate indicates the speed of change, it is expressed as a percentage per unit of time, the number of percentage points that change in one unit of time. In most cases, this unit of time will be taken to be the conversion period. However, within one unit of time there may be more than one conversion. In such a case, the percentage rate applied to one period will be proportionally smaller and termed *factual*, while the percentage rate applied to the entire unit of time will be termed *nominal*. As we can see, the nominal percentage rate defines the speed of change of a population and is related to a corresponding unit of time.

In the case of continuous growth, the population increases (or decreases) without pause, all the time. Therefore, the conversion period is close to zero. However, even in this case the speed of change is expressed as a nominal percentage rate. In many practical problems, especially financial ones, the period of a nominal percentage rate is one year. Unless otherwise noted, in this work we will also consider the period of the nominal percentage rate to be one year.

We shall be analyzing some problems dealing with interest in which there is a constant percentage rate and the population is a function of time. Thus, we shall be deriving regular and compound (cumulative) interest formulas. It will be seen that the regular interest is a linear function of time, while compound interest is exponential.

1.2 Regular Interest

The essence of regular interest is that it is calculated only from the initial population, when the percentage is of a constant size for each unit of time. The rate of growth of regular interest is constant, as equal periods of time correspond to an equal increase in the interest.

Let us write an equation expressing the initial population's dependence on the number of transitions (time) and the interest rate. If S_0 is the initial number, S_n is the final (compounded) number, p is the interest rate, i.e., the percentage calculated for a unit of time, $i = \frac{p}{100}$ is the interest rate expressed as a fraction (a decimal), and n is the time measured in periods, then the increase in the initial number (the initial sum) over *n* periods (the interest) will be $S_0 \cdot i \cdot n$. Adding the increase to the initial sum we get

$$S_n = S_0 + S_0 \cdot i \cdot n .$$

From here we can derive the formula for regular interest (percentages):

$$S_n = S_0 (1 + i \cdot n).$$

This formula has been derived measuring time in periods. Here time is expressed in natural numbers. Therefore the compounded sum S_n changes discretely. However, it is obvious that time may also be measured continuously, i.e. expressed in real numbers. In this case the final number (the compounded sum) shall be written not as S_n but as S, and the formula for regular interest becomes a simple linear function of time.

$$S = S_0 (1 + i \cdot n). \qquad (1.1)$$

The growth of the initial number is calculated with the resulting formula. Occasionally it is necessary to calculate not increase, but decrease. In such a case,

$$S_n = S_0 - S_0 \cdot i \cdot n .$$

Keeping in mind the abovementioned conditions, we get:

$$S = S_0 (1 - i \cdot n). \qquad (1.2)$$

1.3 The Time Factor. Compound (Cumulative) Interest

Compound, as well as regular, interest can be used to calculate the quantitative parameters of certain processes that occur in nature and society and that change over time. The essence of compound interest is that each new population (the contents of a new period: the number of residents or animals, a biological product, a mass of radioactive material, capital, etc.) is calculated by evaluating the change over the previous period (most often an increase: in the number of residents or animals, growth of biomass, interest, etc.). In other words, there is an accumulation of growth: the increase is added to the previous population. The increase in the new period is calculated not only from the initial contents under calculation, but also from the increase of the previous period.

Interest is termed compound (or cumulative) when, at the beginning of every period, the part of the population that increased over the previous period is automatically added to the main population and continues to increase together with it.

It has been determined that the growth rate of such a population is proportional to its size: the larger a population is at a given moment of time, the faster it increases at that moment, and vice versa.

Mathematically, compound interest is repeatedly determined at the end of every period by multiplying the previous value by the coefficient of the speed of change.

When n = 1, 2, 3, ..., we find the following:

$$S_1 = S_0 + S_0 i = S_0 (1 + i),$$

$$S_2 = S_1 + S_1 i = S_1 (1 + i) = S_0 (1 + i)(1 + i) = S_0 (1 + i)^2 ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_n = S_{n-1} + S_{n-1} i = S_0 (1 + i)^n .$$

Therefore the final number (the compounded sum) after *n* periods is as follows:

$$S_n = S_0 (1 + i)^n .$$

In deriving this formula it has been assumed that the resulting increase in the number will be added to the initial number at the end of every period, because time, as in the case of regular interest, is changing discretely. However, the time value can change continuously in this case as well. Then the formula for compound interest is a simple exponential function of time ($n \geq 0$).

$$S = S_0(1+i)^n. \qquad (1.3)$$

If $i > 0$, the resulting expression is an evenly increasing function, while if $-1 < i < 0$, it is an evenly decreasing function. In the latter case the following equation would be more convenient:

$$S = S_0(1-i)^n, \qquad (1.4)$$

where i is a coefficient of decline ($i > 0$).

Example 1.1
The number of residents of town X grows, on average, by 0.7 percent per annum. What will be the increase in the number of town residents over 12 years, if at the present time there are 7.5 million residents?
Solution.
Data: S_0 = 7,500,000; i = 0.007; n = 12.
Using the compound interest formula (1.3), we get:

$$S = 7{,}500{,}000\ (1 + 0.007)^{12} = 8{,}154{,}830;$$
$$S - S_0 = 8{,}154{,}830 - 7{,}500{,}000 = 654{,}830.$$

Answer: over 12 years the number of town residents will increase by 654,830.

Example 1.2
A factory plans to increase production by 75% over 5 years. What should its annual increase in production be?
Solution.
Data: n = 5. An increase in production of 75% means that $S = S_0 \cdot 175\%$ or $\frac{S}{S_0} = 1.75$.
Using the compound interest formula (1.3), we express i:

$$(1+i)^n = \frac{S}{S_0};$$
$$i = \sqrt[n]{\frac{S}{S_0}} - 1.$$
$$i = \sqrt[5]{1.75} - 1 \approx 1.1184 - 1 = 0.1184.$$

Because $p = 100 \cdot i$, the annual increase in production should be 11.84%. It is worth noting that the resulting number is smaller than a proportional part of the percentage (75:5 = 15).
Answer: 11.84%.

Example 1.3
The half-life T (the time during which the initial amount of radioactive material decreases by half) of one of the isotopes of radioactive radium equals five days. What is the decay rate coefficient λ, if the law of decay[1] is expressed by the equation $S = S_0(1-\lambda)^n$?

[1] The law of radioactive decay $S = S_0(1-\lambda)^n$ is more frequently expressed by the equation $S = S_0 \cdot e^{-\lambda \cdot t}$. In general, if the decay rate coefficients λ used are equivalent, the results are the same.

Solution.
It was stated that the quantity of radium decreases by one half over the given time (half-life). That means that $S = 0.5 \cdot S_0$. If we incorporate this expression into the given equation, we get $0.5 \cdot S_0 = S_0(1-\lambda)^5$. From this we find that $\lambda = 1 - \sqrt[5]{0.5} = 0.129$.
Answer: the decay rate (measured in days) coefficient λ = 0.129.

1.4 Continuous Exponential Change

Let us now return to the original (discrete) compound interest formula in which each period's increase is added to the earlier population at the end of that period.

Obviously the period's increase can be added to the main population not only once, at the end of the period, but also more frequently.

It has been mentioned that the time between two consecutive interest calculations is called the conversion period. An interest rate is usually determined for a unit of time. However, a unit of time can have not only one, but several conversion periods.

An interest rate calculated for a unit of time containing several conversion periods is called a *nominal interest rate.*

We shall be using this interest rate most often in our calculations. It should be noted that even when additional calculations are not performed, that is, when there is only one calculation for a unit of time, the interest rate can still be called nominal.

We have already noticed that the unit of time in many practical problems is one year, though in practice it could be different. As the unit of time can have not only one, but several conversion periods, the interest rate for one period will equal the nominal interest rate divided by the number of conversion periods. The period's interest rate is the factual interest rate for that period.

Thus, if in one unit of time (let us say one year) we should perform not one, but several calculations (i.e., if we should add the increase to the earlier population at the end of every part $1/k$ of the unit of time—let us say, every month) while the interest rate, as before, remained nominal, then population growth would be as follows:

$$S_{1/k} = S_0\left(1+\frac{i}{k}\right),$$

$$S_{2/k} = S_0\left(1+\frac{i}{k}\right) + S_0\left(1+\frac{i}{k}\right)\frac{i}{k} = S_0\left(1+\frac{i}{k}\right)\left(1+\frac{i}{k}\right) = S_0\left(1+\frac{i}{k}\right)^2,$$

....

$$S_{k/k} = S_0\left(1+\frac{i}{k}\right)^k.$$

So, at the end of the first unit of time (after one year) the size of the population will be:

$$S_1 = S_0\left(1+\frac{i}{k}\right)^k$$

Continuing to reason in this way, it is easy to see that at the end of two units of time (after two years) the size of the population will be:

$$S_2 = S_0\left(1+\frac{i}{k}\right)^{2k}.$$

Finally, after n analogical units of time (after n years) we will have:

$$S_n = S_0\left(1+\frac{i}{k}\right)^{nk}.$$

If we assume that n can acquire not only whole number values, we will have:

$$S = S_0\left(1+\frac{i}{k}\right)^{nk} \qquad (1.5)$$

We have now derived a formula for compound interest in which there are several interest conversions in one unit of time.

Example 1.4

Calculate the value of €1 invested at the nominal (annual) interest rate of 8% after one, five, and ten years, when the conversion period is: 1) one year; 2) half a year; 3) one quarter; 4) one month; 5) one day.

Solution.

Data: $S_0 = 1$; $i = 0.08$; a) $n = 1$, b) $n = 5$, c) $n = 10$; 1) $k = 1$, 2) $k = 2$, 3) $k = 4$, 4) $k = 12$, 5) $k = 365$.

When $n = k = 1$ we have a typical interest situation:

$S = 1\cdot(1+0.08) = 1.08$ (€).

When $n = 1$; $k = 2$ we will have:

$$S = 1\cdot\left(1+\frac{0.08}{2}\right)^{2} = 1.0816 \text{ (€)}.$$

When $n = 1$; $k = 4$ we will have:

$$S = 1\cdot\left(1+\frac{0.08}{4}\right)^{1\cdot 4} = 1.082432 \text{ (€)}.$$

The calculations for the other values of n and k are performed in a similar way. Finally, when $n = 10$; $k = 365$, we will have:

$$S = 1\cdot\left(1+\frac{0.08}{365}\right)^{10\cdot 365} = 2.225346 \text{ (€)}.$$

Let us compile a table of the results of all the calculations:

No.	Conversion period	Conversion frequency k	Value of investment (€) at the end of		
			1 year	5 years	10 years
1	One year	1	1.0800	1.4693	2.1589
2	Half a year	2	1.0816	1.4802	2.1911
3	One quarter	4	1.0824	1.4859	2.2080
4	One month	12	1.0830	1.4898	2.2196
5	One day	365	1.0833	1.4918	2.2253

The results indicate that, even though the nominal interest rate is constant, the value (accumulated sum) of the investment at the end of the given time period changes as the conversion frequency is changed. This raises the problem of interest rate equivalency. We shall explore equivalency problems in greater detail in the next chapter.

Analysis of the example results shows that, as the conversion frequency k is increased, the accumulated sum at the end of the time period grows. It should be noted that the growth rate decreases for larger values of k, and the sequence of the values approaches a certain limit (*Figure 1.1*). In addition, we see that the dependence of the accumulated amount on the conversion frequency is not very large: even when the number of conversions is significantly increased, if the growth rate is 8%, the increase in the accumulated sum does not exceed 3%. Moreover, it can be seen from the example that only the initial increase of conversion has a strong effect on the increase in the accumulated sum: with an annual accumulation rate it is not very effective to convert more often than once a month.

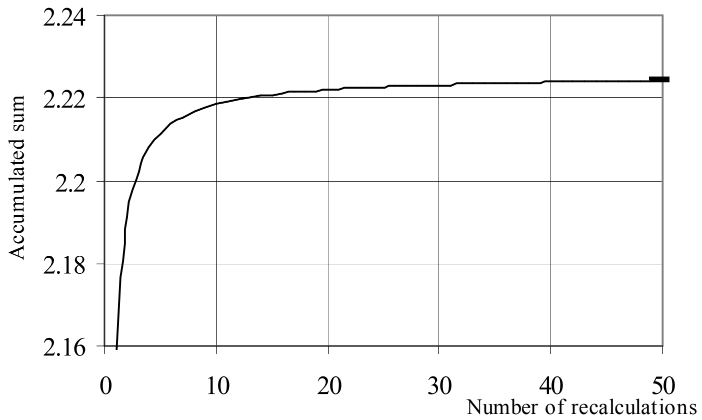


Fig. 1.1 Relation of sum accumulated over 10 years to number of recalculations in one period (K_0 = 1; i = 0.08)

Now let us return to the formula for compound interest (1.5), when there are several interest conversions k in one unit of time. If we assume that the increase is continuous, i.e. if $n \to \infty$ and the duration of the part of the period approaches 0, $\left(\frac{1}{k} \to 0\right)$, then

$$S = S_0 \lim_{k\to\infty}\left(1+\frac{i}{k}\right)^{k\cdot n} = S_0\left(\lim_{k\to\infty}\left(1+\frac{i}{k}\right)^{\frac{k}{i}}\right)^{\frac{i}{k}\cdot k\cdot n}.$$

Because $\lim_{k\to\infty}\left(1+\frac{i}{k}\right)^{\frac{k}{i}} = e \approx 2{,}718\ldots$, and also $\frac{i}{k}kn = i\cdot n$, then

$$S = S_0 e^{i\cdot n}. \qquad (1.6)$$

We have now derived an equation for continuous (natural) exponential change, sometimes called the equation of continuous compound interest. It can be used to calculate the quantitative parameters of various processes: an increase in the number of residents, the growth of timber in a forest, radioactive decay, the multiplication of bacteria during a biochemical process, etc. If $i > 0$ we have exponential growth, and when $i < 0$ we have exponential decline.

Example 1.5 (a continuation of Example 1.4)
Calculate the value of €1, invested with an 8% nominal (annual) interest rate, after 1, 5, and 10 years, using the equation for continuous change.
Solution.
Data: S_0 = 1, i = 0.8, a) n = 1; b) n = 5; c) n = 10.

$$S = e^{0,08} = 1.0832 \ (€),$$

$$S = e^{0,08\cdot 5} = 1.4918 \ (€),$$

$$S = e^{0,08\cdot 10} = 2.2255 \ (€).$$

Answer: a) €1.0832; b) €1.4918; c) €2.2255

If we compare the results of the last two examples, we see that daily recalculation and continuous change give very similar results.

Example 1.6

The half-life T of the isotope E of radioactive radium equals five days. What is the decay coefficient λ for this material if the law of decay is expressed by the equation $S = S_0 e^{-\lambda \cdot n}$?

Solution.

It was stated that the quantity of radium decreases by one half over the given time (half-life). That means that $S = 0.5 \cdot S_0$. It was also stated that $n = 5$. Inserting these values into equation (1.6), we get

$$\frac{S_0}{2} = S_0 e^{\lambda \cdot 5}; \ \lambda = \frac{\ln 0.5}{5} = 0.139 \cdot$$

Answer: the decay coefficient λ = 0.139.

If we compare this result with that of Example 1.3, where λ = 0.129, we see a certain disagreement. It arises from the different frequencies of conversion. Let us note that the decay half-lives in both cases are the same. This means that only when the decay coefficients are equivalent will the results obtained be the same.

It should be emphasized that the formula for natural change is not widely used in economical calculations. However, it is useful for evaluating other populations and is therefore rather common.

1.5 Equivalent Interest Rate

In the previous section, when discussing the fact that there can be several conversions in one unit of time, we encountered the problem of interest rate equivalence.

In finance, one often solves tasks in which investments are compared. In these cases the values of percentage rates need to be set in such a way that they give equivalent results even under different conditions of interest capitalization. For example, consider this problem of compound interest: if, given percentage rate i, €1 gives P interest in one year, then what should be percentage rate $i_{12} = i^{(k)}/12$ so that, capitalizing interest monthly, the same interest P would be returned after one year?

Equivalent to percentage rate i is the percentage rate i_k ($i_k = i^{(k)}/k$), which, when interest is capitalized k times per year, yields the same interest in the same time.

To better understand the problem of equivalence, let us analyze a problem similar to *Example 1.4*. Now, however, in formulas and elsewhere we will use K (for capital) rather than S (for number).

Example 1.7

€1 is deposited in a bank account for a period of one year when the nominal annual compound interest rate is 12%. Calculate the value of the deposit after one year if in that year it is capitalized one, two, four, and twelve times.

Solution.

Data: $K_0 = 1$; $i^{(k)} = 0.12$; $k = 1$; $k = 2$; $k = 4$; $k = 12$.

The value of the deposit after one year, capitalized once, will be:

$$K = 1 \cdot (1 + 0.12) = 1.12 \ (€).$$

Obviously, the deposit increases 12% in one year.

The value of the deposit after one year, capitalized twice, will be:

$$K = 1 \cdot \left(1 + \frac{0.12}{2}\right)^2 = 1.1236\,(€).$$

We note that in a year the deposit increases 12.36 %. We also note that this increase is larger than the previous one. The difference between the increases is 0.36%.

The value of the deposit after one year, capitalized four times, will be:

$$K = 1 \cdot \left(1 + \frac{0.12}{4}\right)^4 = 1.1255\ \ (€).$$

In a year the deposit increases 12.55%.

The value of the deposit after one year, capitalized every month, will be:

$$K = 1 \cdot \left(1 + \frac{0.12}{12}\right)^{12} = 1.1268\,(€).$$

In this case, in one year the deposit increases 12.68%.

Let us compile a table of the results:

No.	Conversion period	Conversion frequency k	Nominal interest rate $i^{(k)}$ (%)	Factual (period) interest rate $i^{(k)}/k$ (%)	Value of investment after one year (€)	% increase per year
1	One year	1	12	12	1.12	12
2	Half a year	2	12	6	1.1236	12.36
3	Quarter	4	12	3	1.1255	12.55
4	Month	12	12	1	1.1268	12.68

We see that even when the nominal annual interest rate is 12% but it is converted 2, 4, or 12 times a year, the increase per year exceeds 12% and is correspondingly equal to 12.36, 12.55 or 12.68%. Thus we may say that 6% half-yearly interest is equivalent to 12.36% annual, 3% quarterly interest is equivalent to 12.55% annual, and 1% monthly interest is equivalent to 12.68% annual.

Let us analyze one more example.

Example 1.8

€5,000 are deposited in a bank account for one year with an annual compound interest rate of 12%. What will be the half-yearly, quarterly and monthly compound interest rates equivalent to it?

Solution.

The annual interest rate $i^{(k)} = 0.12$. The interest rates for half a year ($i_2 = i^{(2)}/2$), a quarter ($i_4 = i^{(4)}/4$), and a month ($i_{12} = i^{(12)}/12$) are considered to be a corresponding part of the nominal interest rate. In addition, the capitalization period for this interest is correspondingly half a year, one quarter or one month.

If the interest is calculated once a year, then at the end of the year the value K of the initial deposit will be:

$$K = 5000(1 + i) = 5000 \cdot 1.12 = 5600\,(€).$$

The value of the deposit after one year will be the same when the period of interest capitalization is half a year:

$$5000\left(1 + \frac{i^{(2)}}{2}\right)^2 = 5600\,(€),$$

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.12\,,$$

$$i^{(2)} = 2\left(\sqrt{1.12}-1\right) \approx 0.1166\,.$$

We have determined that when the interest is capitalized twice a year, the nominal interest rate equals 11.66%, which is less than the annual interest rate (12%).

We shall now similarly calculate the quarterly interest rate under the initial conditions:

$$\left(1+\frac{i^{(4)}}{4}\right)^4 = 1.12\,,$$

$$i^{(4)} = 4\left(\sqrt[4]{1.12}-1\right) \approx 0.1148\,,$$

and also the monthly interest rate:

$$\left(1+\frac{i^{(12)}}{12}\right)^{12} = 1.12\,,$$

$$i^{(12)} = 12\left(\sqrt[12]{1.12}-1\right) \approx 0.1138\,.$$

We may note that the results do not depend on the size of the deposit.

Let us compile a table of results:

Period	Number of periods k per year	Nominal interest rate $i^{(k)}$ (%)	Factual (period) interest rate $i^{(k)}/k$ (%)
Year	1	12	12
Half year	2	11.66	5.83
Quarter	4	11.49	2.87
Month	12	11.39	0.95

It is important to clearly understand which of these interest rates are equivalent to each other. In this example, the 12% nominal annual interest rate is equivalent to the nominal interest rates of 11.66%, 11.48%, and 11.40% when their conversion periods are correspondingly 6, 3 and 1 month. Equivalency may also be estimated as follows: an annual interest rate of 12% is equivalent to a half-yearly rate of 5.83%, a quarterly rate of 2.87%, and a monthly rate of 0.95%. Finally, the following interest rates will also be equivalent: a monthly interest rate of 0.95% is equivalent to an annual rate of 11.66% compounded every 6 months, or to an annual rate of 11.48% compounded every 3 months, and so on.

In solving the problem, some operations were repeated. Now let us generalize our calculations. If i, $i^{(2)}$, $i^{(4)}$, $i^{(12)}$, ..., $i^{(k)}$ are equivalent nominal annual interest rates whose conversion periods are, respectively, one year, half a year, one quarter, and one month, etc., then

$$1+i = \left(1+\frac{i^{(2)}}{2}\right)^2 = \left(1+\frac{i^{(4)}}{4}\right)^4 = \left(1+\frac{i^{(12)}}{12}\right)^{12} = \ldots = \left(1+\frac{i^{(k)}}{k}\right)^k .$$

We may therefore derive

$$1+i = \left(1+\frac{i^{(k)}}{k}\right)^k ;\ 1+\frac{i^{(k)}}{k} = \sqrt[k]{1+i}\ ;\ \frac{i^{(k)}}{k} = \sqrt[k]{1+i}-1\,.$$

Thus we get an equation for the equivalency of nominal interest rates:

$$i^{(k)} = \left(\sqrt[k]{1+i}-1\right)k\,, \qquad (1.7)$$

where k is the number of conversions per year, $i^{(k)}$ is the nominal interest rate with k conversions per year, and i is the nominal annual interest rate ($k = 1$).

Solving the equation for interest rate i, we find that

$$i = \left(1 + \frac{i^{(k)}}{k}\right)^k - 1. \qquad (1.8)$$

Example 1.9
A deposit can be made to one of two banks. The first bank is offering 14% annual interest, the second 1.1% monthly interest (compound percentages). Which bank is offering larger interest?
Solution.
Data: i =14; $\frac{i^{(12)}}{12} = 0.011$; k =12.

Inserting the factual (period) interest rate $\frac{i^{(12)}}{12} = 0.011$ into formula (1.8) we will derive the annual interest rate:

$i = (1 + 0.011)^{12} - 1 = 0.1403$

The derived interest rate of 14.03% is larger than the interest rate offered by the first bank, namely 14%. Therefore we may conclude that the interest offered by the second bank is larger.

This problem may also be solved using formula (1.7). We take i = 0.14 and find an equivalent nominal interest rate with 12 conversions.

$i^{(12)} = \left(\sqrt[12]{1 + 0.14} - 1\right)12$ =0.1317.

From here we can find the monthly interest rate. It is $\frac{i^{(12)}}{12} = 0.01098$. The calculated monthly interest rate of 1.098% is smaller than the interest offered by the second bank (1.1%). Thus we reach the same conclusion again: the interest offered by the second bank is larger.
Answer: the second bank is offering larger interest.

Formulas (1.7) and (1.8) are useful in determining the equivalency of two nominal interest rates. In order to determine the equivalency of a nominal and a factual interest rate, it is more convenient to express the period interest rate separately. By marking $\frac{i^{(k)}}{k} = i_k$, we get the following from the above formulas:

$$i_k = \sqrt[k]{1 + i} - 1, \qquad (1.9)$$

or

$$i = (1 + i_k)^k - 1. \qquad (1.10)$$

Let us solve Example 1.9 using formula (1.9):

$i_{12} = \sqrt[12]{1 + 0.14} - 1 \approx 0.01098$.

This problem can also be solved using formula (1.10). Then

$i = (1 + 0.011)^{12} - 1 \approx 0.1403$.

Conclusion: because 1.098 < 1.1 (or because 14.03 > 14), the interest rate offered by the second bank is, however insignificantly, larger.

Example 1.10
A bank is paying 5% annual interest (commercial[2] interest: “30/360”), however the interest is paid every month. What is the equivalent annual interest rate if we assume that the interest is reinvested immediately after being paid?
Solution.

Compound interest may also be applied in this case. Using formula (1.8), we get:

$$i = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 0.0512.$$

The result is that $i = 5.12\%$.
Conclusion: the equivalent interest rate is larger than the base rate by 0.12%.

Now we shall compare interest rates based on compound percentages and the law of natural growth. We will assume that both types of calculations should yield identical results. Let us make the right sides of the equations $K_n = K_0(1+i)^n$ ir $K_n = K_0 e^{n \cdot i^{(k)}}$ equal:

$$K_0(1+i)^n = K_0 e^{n \cdot i^{(k)}}.$$

Cancelling out and calculating the n^{th} root, we get

$$1 + i = e^{i^{(k)}},$$

$$i^{(k)} = \ln(1+i). \qquad (1.11)$$

Example 1.11
A compound percentage rate equals 10%. What is the equivalent rate of continuous compound interest?
Solution.
Using *formula (1.11)* we find:

$$i^{(k)} = \ln 1.1 = 0.0953.$$

We have determined that 10% compound interest and 9.53% natural growth law give the same results over the same period.
Answer: 9.53%.

1.6 Discounting Using Compound Percentages

Discounting is the recalculation of any predetermined meaning of a value for an earlier period.

In the case of regular interest, in the discussion about bank discounting, we needed to recalculate the final value into an earlier, interim one. With compound interest, various interim values are usually recalculated for the initial (the present) time, that is, the present value of the capital is determined. In this case, the equation for compound interest is solved to express capital K_0:

$$K_0 = \frac{K}{(1+i)^n}.$$

Having marked $1 + i = r$, o $\frac{1}{r} = v$, we find:

$$K_0 = K \cdot v^n.$$

[2] Commercial interest “30/360” means that the calculations are carried out for a conditional month (30 days) and a conditional year (360 days).

We have thus determined a formula for discounting compound interest. Here v is a discount coefficient. From this formula we can see that the present value K_0 of monetary funds equals the product of the discounted funds K and the discount coefficient v raised to the power n of the number of years.

Example 1.12

What sum of money needs to be deposited in a savings bank so that, given a 6% annual compound interest rate, there would be €9,000 after 5 years?

Solution.

We discount the given sum of money:

$$K_0 = 9000\left(\frac{1}{1+0.06}\right)^5 \approx 9000 \cdot 0.74726 = 6725.32 \text{ (€)}.$$

Answer: €6,725.32.

Example 1.13

There are two ways to purchase some computer equipment: first, to buy one device with a lifetime of 6 years for €10,000; second, to buy two devices with lifetimes of 3 years each for €5,200 each, buying one of them now and the other when the first one wears down in 3 years. Let us determine which way is more economical, assuming an annual interest rate of 8% and keeping all other conditions the same in both cases.

Solution.

It may appear at first glance that the first option is more effective. However, when the present value of the expenses of the second option is determined, that is, when we discount the funds for implementing the second option, we see that:

$$5\,200 + \frac{5\,200}{(1+0.08)^3} \approx 5\,200 + 4127.93 = 9327.93 \text{ (€)}.$$

As we have discovered, the second way is more effective, since its present value is smaller by €672.07 (10,000 – 9,327.93), or approximately 6.7% of the present value of the first project.

Example 1.14

There are two options (two projects) for the purchase and maintenance of a workbench, and their returns are identical. Let us determine which one of the two investment options is more effective, if the annual interest rate is 8% and the following purchase and maintenance expenses at the beginning of every year are known:

Years	Option I	Option II
1	2,000	4,400
2	400	100
3	600	200
4	2,000	300
5	400	400
6	600	500
Total:	6,000	5,900

Solution.

Let us determine the present value of each project by discounting the expenses of every year for the initial moment. After discounting the expenses needed to purchase and maintain the workbench, we get:

Option I

$$2\,000+\frac{400}{1{,}08}+\frac{600}{1{,}08^2}+\frac{2\,000}{1{,}08^3}+\frac{400}{1{,}08^4}+\frac{600}{1{,}08^5}\approx$$

$$\approx 2000+370{,}37+514{,}40+1587{,}66+294{,}01+408{,}35=5174{,}8.$$

Option II

$$4\,400+\frac{100}{1{,}08}+\frac{200}{1{,}08^2}+\frac{300}{1{,}08^3}+\frac{400}{1{,}08^4}+\frac{500}{1{,}08^5}\approx$$

$$\approx 4400+92{,}59+171{,}47+238{,}15+294{,}01+340{,}29=5536{,}51.$$

The calculations of the present value indicate that the second project is more expensive by €361.71 (5,536.51 – 5,174.8).

When investment calculations are performed using Microsoft Excel, the financial NPV function can be used. Using this function, the following commands can be written down to solve Example 1.14:

"=NPV(8%;400;600;2000;400;600)+2000 equals €5174.80."

"=NPV(8%;100;200;300;400;500)+4400 equals €5536.51."

Discounting problems will be further analyzed in subsequent chapters.

1.7 The Value Equation

In an earlier *Section (1.6)* we discussed the equivalence of interest rates. Now we shall discuss the equivalence of financial obligations. Usually, two interested parties participate in financial transactions, and for that reason the transaction is possible only when the interests of both parties involved are balanced. Because financial obligations are usually connected to certain time periods, it is necessary to determine the dependence of the value of money on time. As we have seen several times, the value of money changes with time: €1,000 now and one year from now are not the same money: if we deposit it in a bank, the €1,000 we have today will in one year produce interest, as well.

In order to compare sums of money that are received or paid out at different times, it is necessary to recalculate them for the same moment in time, which is called the *moment (*or *point) of comparison*. Either future or present money values can be used in recalculation.

The principle of equivalence of financial obligations demands that, for example, credit taken out now but paid back after a certain term be covered by an equivalent sum. This sum is dependent on the amount of time that passes from the taking of the credit to its coverage, as well as on the size of the interest rate agreed on. Equivalence of obligations is mathematically expressed by an equivalence or *value equation*. Here it is important to emphasize that the value equation can be written down not only for the initial time, but also for any other moment of comparison. Equations written down in such a way are equivalent, i.e., they have the same solutions.

The value equation is very simple if the credit is covered in one instalment. Discounting the covering instalment, we find:

$$K=\frac{x}{(1+i)^n},$$

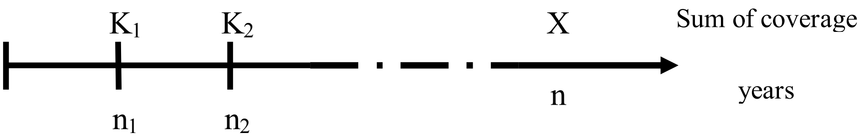
where K is credit given for n years with an interest rate of i, and x is the instalment that covers it.

We can easily find the covering sum with this equation:

$x = K(1 + i)^n$.

Obviously, this equation is the formula for compound percentages.

Now let us say that the credit is covered not all at once, but in several instalments.



The interim sums K_1, K_2, ..., are paid after n_i (i = 1, 2, ...) terms have passed. We must determine the value of the final instalment X, to be paid after n years. By discounting the coverage instalments for the initial moment, that is, the moment when the credit was taken, we get the value equation:

$$K = \frac{K_1}{(1+i)^{n_1}} + \frac{K_2}{(1+i)^{n_2}} + ... + \frac{X}{(1+i)^n} \ . \qquad (1.12)$$

Let us illustrate our discussion with examples.

Example 1.15
A loan of €10,000 is taken out for 3 years with an 8% annual (compound) interest rate. The loan is to be repaid as follows: €8,000 will be returned after 2 years, and the remainder at the end of the term in 3 years. Let us determine the size of the final instalment.
Solution.

We will make an equation assuming that the size of the loan and the returned sum of money are equivalent. For that purpose we discount the repaid sum of money back to the moment at which the loan was taken out. We get the equation:

$$10\,000 = \frac{8\,000}{1.08^2} + \frac{X}{1.08^3} \ .$$

Having multiplied both sides of this equation by 1.08^3 and solved it for x, we get an equivalent equation:

$$X = 10\,000 \cdot 1.08^3 - 8\,000 \cdot 1.08 \ .$$

It can be seen that the new equation is also a value equation, written however for a different moment of comparison: after the term of repayment. With similar rearrangements we can find a value equation for any moment of comparison.
From any such equation we find that X = €3,957.12. This is the answer.

An analogous value equation can also be made when there is not one, but several interim instalments.

Example 1.16
A loan of €10,000 is taken out for 5 years with a 6% annual interest rate. The loan is to be repaid as follows: €4,000 will be returned after 2 years, €5,000 after 4 years, and the remainder at the end of the term in 5 years. Let us determine the size of the final instalment.
Solution.

Let us make a value equation by discounting the repaid sums to the moment the loan was taken out:

$$10\,000 = \frac{4\,000}{1.06^2} + \frac{5000}{1.06^4} + \frac{X}{1.06^5} \ .$$

By performing a similar rearrangement as in *Example 1.15*, we get:

$$X = 10\,000 \cdot 1.06^5 - 4\,000 \cdot 1.06^3 - 5\,000 \cdot 1.06 \ .$$

Noting that the latter equation is the value equation written for the moment of the end of the repayment term, we find the size of the final instalment: X = €3,318.19. This is the answer.